1. Adapting the proof for the Shannon coding problem to show that the encoding function f can be chosen to be a linear function, i.e. $f(x) + f(y) = f(x + y)$ for any $x, y \in \{0, 1\}^n$. Briefly explain how this makes the encoding process easier.

2. A planar graph is a graph that can be drawn on the plane such that the edges do not cross with each other. One can use the Euler formula to prove that any simple planar graph with n vertices can have at most $3n - 6$ edges. A graph G has *crossing number* k if k is the maximum number such that any drawing of G on the plane has at least k pairs of edges crossing. (So, by the above result, the crossing number of any simple graph with at least $3n-5$ edges is at least one.) Prove that the crossing number of any simple graph with at least $m \geq 4n$ edges is at least $\frac{m^3}{64n^2}$. (Hint: random sampling.)

3. (Optional, since its not about probabilistic methods.) Prove that any 2-coloring of the edges of a complete graph with at least 2^{2k} vertices must have a monochromatic clique of size k.

4. (Optional, since its not about probabilistic methods.) In this exercise we will construct a triangle-free graph with arbitrarily high chromatic number. Let G_2 be a graph consisting of a single edge. Given $G_n = (V, E)$, construct G_{n+1} as follows. The new set of vertices is $V \cup V' \cup \{z\}$, where V' is a copy of V and z is a single new vertex. $G_{n+1}[V]$ is isomorphic to G_n. For each vertex $v' \in V'$ which is a copy of $v \in V$, we connect it by edges to all vertices $w \in V$ such that $(v, w) \in E$. We also connect each $v' \in V'$ to the new vertex z. Prove that G_n is triangle-free and $\chi(G_n) = n$.